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Unraveling Nonlinear Neutral Dynamics: A Study of Controllability in Weak Conditions

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Abstract

This research investigates a particular category of second-order stochastic differential equations driven by Q-Wiener processes. We make significant contributions by demonstrating results regarding existence and uniqueness under a Weak condition, which is less stringent than the conventional Lipschitz criterion. Additionally, we outline conditions that guarantee the controllability of the mild solution, utilizing the theory of stopping times.

1 Introduction

In this paper, we delve into the approximate controllability of a specific second-order stochastic differential equation, formulated as follows:

$$\begin{cases} \mathrm{d}(Z'(\tau) - f_1(\tau, Z(\tau), U(\tau)) = [AZ(\tau) + BU(\tau) + f_2(\tau, Z(\tau), U(\tau))]\mathrm{d}\tau + f_3(\tau, Z(\tau), U(\tau))\mathrm{d}B(\tau), \\ Z(0) = z_0, \quad Z'(0) = z_{00}, \quad \tau \in [0, T], \end{cases} \tag{1}$$

set within the context of a real separable Hilbert space H_2. Here, B represents a given H_1-valued Wiener process characterized by a positive nuclear covariance operator Q, defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with $\mathrm{Trace}(Q) < \infty$. The process is further equipped with a normal filtration $\{\mathcal{F}_\tau\}_{\tau \geq 0}$ generated by B. The linear operator $A : D(A) \subset \mathrm{H}_2 \to \mathrm{H}_2$ is a closed linear operator that generates a strongly cosine-family in H_2. Additionally, $B : \mathrm{H}_3 \to \mathrm{H}_2$ is a bounded linear operator, while the functions $f_1, f_2 : [0, T] \times \mathrm{H}_2 \times \mathrm{H}_3 \to \mathrm{H}_2$ and $f_3 : [0, T] \times \mathrm{H}_2 \times \mathrm{H}_3 \to \mathcal{L}^2(\mathrm{H}_0, \mathrm{H}_2)$ are measurable mappings. Here, H_i denotes a real separable Hilbert space, and $\mathcal{L}^0 = \mathcal{L}^2(\mathrm{H}_0, \mathrm{H}_2) = \mathcal{L}^2(Q^{1/2}\mathrm{H}_1, \mathrm{H}_2)$ is the space of bounded, linear, Q-Hilbert–Schmidt operators from H_0 to H_2. Notably, η_0 and ξ_0 are $\mathcal{F}_0$-measurable random variables in H_2 that are independent of B and have finite moments of order $p \geq 2$, while $U : \Omega \times [0, T] \to \mathrm{H}_3$ serves as the control function.

We define the concept of a 'Mild Solution' for the equation (1), as detailed in Section 2.

Definition 1.1 *A stochastic process $X \in C^p([0, T], \mathrm{H}_2)$ is classified as a mild solution of equation (1) if, for any $U \in C^p([0, T], \mathrm{H}_3)$, it satisfies the following stochastic integral equation almost surely:*

$$\begin{aligned} Z(\tau) = C(\tau)Z(0) + S(\tau)(Z'(0) - f_1(0, Z(0), U(0))) + \int_0^\tau S(\tau - v)BU(v)\mathrm{d}v \\ + \int_0^\tau C(\tau - v)f_1(v, Z(v), U(v))\mathrm{d}v + \int_0^\tau S(\tau - v)f_2(v, Z(v), U(v))\mathrm{d}v \\ + \int_0^\tau S(\tau - v)f_3(v, Z(v), U(v))\mathrm{d}B(v). \end{aligned}$$

In recent years, there has been a surge of interest in control problems framed as abstract differential equations [4, 3]. Stochastic systems provide a robust framework for modeling and analyzing phenomena such as population dynamics, financial market behavior, and thermal processes in materials with memory.

Modern control theory emphasizes the significance of controllability, which serves as a foundation for addressing challenges in both deterministic and stochastic environments. Calman's pioneering work on controllability in the 1960s set the stage for subsequent advancements in the field. Subsequent studies have explored various dimensions of controllability, including exact controllability [7, 8], optimal control [6], and approximate controllability [6, 2].

Numerous studies have turned to Gronwall's inequality as a means to establish the existence and uniqueness of solutions. However, the importance of finite factors is often neglected. While some credible sources apply rigorous methodologies, many researchers ultimately rely on Gronwall's inequality, which can create challenges. Specifically, continuity of the solution can only be assured if it remains bounded. The literature suggests [1] that a prudent strategy is to halt the process when solutions reach elevated values to mitigate this issue. In our approach, we also advocate for bounding these factors and incorporate stopping times to preserve generality. This aspect is crucial since the Lipschitz conditions serve as a specific case of our investigation, making the Gronwall lemma particularly relevant in this context.

2 Preliminaries and Notations

In this section, we establish a foundational framework encompassing three separable Hilbert spaces: H_1, H_2, and H_3. We leverage essential concepts from previous studies to introduce Gaussian stochastic processes and Q-Wiener processes.

A stochastic process defined in H_2 is considered Gaussian if it follows a Gaussian distribution across any selection of time points [1]. On the other hand, a Q-Wiener process in H_1 possesses specific characteristics related to its trajectories and increments, governed by a nonnegative trace class operator Q.

Let us introduce the following definitions under the condition $p \geq 2$:

- $(\Omega, \mathcal{F}, (\mathcal{F}_\tau)_{\tau\in[0,T]}, \mathbb{P})$: This notation signifies a complete probability space accompanied by a normal filtration generated by B, with $\mathcal{F} = \mathcal{F}_T$.
- $L^p(\Omega, \mathcal{F}_T, \mathbb{P}, \mathrm{H}_2)$: This denotes the Banach space comprising all stochastic processes $Z : \Omega \times [0, T] \to \mathrm{H}_2$ such that, for every $\tau \in [0, T]$, the random variable $Z(\tau)$ is $\mathcal{F}_T$-measurable, and $\mathbb{E}\, \|Z(\tau)\|^p_{\mathrm{H}_2} < +\infty$.
- $C^p([0, T], \mathrm{H}_2)$: This represents the Banach space of progressively measurable and continuous stochastic processes $Z : \Omega \times [0, T] \to \mathrm{H}_2$ for which $\mathbb{E}\left(\sup_{\tau\in[0,T]} \|Z(\tau)\|^p_{\mathrm{H}_2}\right) < +\infty$.

Cosine Families and Second-Order Cauchy Problems

In this section, we investigate the concept of cosine families and their association with second-order Cauchy problems, drawing a parallel to the relationship between C_0-semigroups and first-order Cauchy problems. For an in-depth exploration of continuous cosine and sine families, refer to [5, 9].

Definition 2.1 *A strongly continuous cosine family $\{C(\tau)\}_{\tau\in\mathbb{R}} \subset \mathcal{L}(\mathrm{H}_2)$ is characterized by the following properties:*

1. *Chapman-Kolmogorov Equation: $C(\tau + v) = 2C(\tau)C(v) - C(\tau - v)$ for all $\tau, v \in \mathbb{R}$.*
2. *Identity Property: $C(0) = I$, where I denotes the identity operator.*

3. *Strong Continuity: The mapping $C(\tau)y$ is strongly continuous in τ for every $y \in \mathrm{H}_2$.*

The associated *strongly continuous sine family* $\{S(\tau) = \int_0^\tau C(v)\mathrm{d}v : \tau \in \mathbb{R}\} \subset \mathcal{L}(\mathrm{H}_2)$ meets the condition $S(0) = 0$ and displays strong continuity. The *generator* of a cosine family is represented as $(A, D(A))$ and is defined by $Ay = \lim_{\alpha \to 0^+} 2\alpha^{-2}(C(\alpha) - I)$ for $y \in D(A)$, where $D(A)$ indicates the domain of A.

According to the Hille-Yosida theorem, a strongly continuous cosine family can be generated by an operator $(A, D(A))$ under necessary and sufficient conditions.

We also summarize several crucial properties of cosine families, including their boundedness, their interplay with corresponding sine families, and additional attributes associated with the generator A.

3 Key Finding

First we define the stochastic analogue of complete controllability and approximate controllability concepts. Now let us introduce the following operators and sets

1. The operator $\mathfrak{L}_0^T : C^p([0,T], \mathrm{H}_3) \to \mathrm{L}^p(\Omega, \mathcal{F}_T, \mathbb{P}, \mathrm{H}_2)$ is defined by

$$\mathfrak{L}_0^T U = \int_0^T S(T-v)BU(v)\mathrm{d}v$$

The operator $\mathfrak{M}_0^T : \mathrm{L}^p(\Omega, \mathcal{F}_T, \mathbb{P}, \mathrm{H}_2) \to C^p([0,T], \mathrm{H}_3)$ is defined by

$$\mathfrak{M}_0^T h = B^* S^*(T-\tau)\mathbb{E}(h|\mathcal{F}_\tau).$$

The controllability Gramian operator is defined by

$$\Lambda_0^T h = \int_0^T S(T-v)BB^*S^*(T-v)\mathbb{E}(h|\mathcal{F}_v)\mathrm{d}v$$

2. The operator $\Gamma_{u,\tau}^T \in \mathcal{L}(\mathrm{H}_2, \mathrm{H}_3)$

$$\Gamma_{u,\tau}^T y = \int_u^\tau S(\tau - v)BB^*S^*(T-v)y\mathrm{d}v,$$

The controllability operator $\Gamma_{u,T}^T \in \mathcal{L}(\mathrm{H}_2, \mathrm{H}_3)$

$$\Gamma_{u,T}^T y = \int_u^T S(T-v)BB^*S^*(T-v)y\mathrm{d}v, \quad 0 \le \tau < T.$$

Definition 3.1 *The system described by Equation* (1) *is termed approximately controllable over the interval* $[0,T]$ *if the range space* $\bar{\mathcal{R}}(T)$ *coincides with* $\mathrm{L}^p(\Omega, \mathcal{F}_T, \mathbb{P}, \mathrm{H}_2)$. *If* $\mathcal{R}(T)$ *equals* $\mathrm{L}^p_{\mathrm{H}_2}(\Omega, \mathcal{F}_T, \mathbb{P})$, *the system is termed exactly controllable. Here,* $\mathcal{R}(T)$ *is defined as the set*

$$\{Z(T, Z(0), Z'(0), U) : U \in C^p([0,T], \mathrm{H}_3)\}.$$

To formulate and prove our main results, we require the following assumptions. Let $\mathcal{B}(\mathrm{H}_i)$ represent the Borel sigma-algebra on the space H_i, while $\mathcal{P}$ denotes progressively measurable σ-fields on $\Omega \times [0,T]$.

(A0) The mappings $C(v), S(v) : \mathrm{H}_2 \to \mathrm{H}_2$ form strongly continuous Cosine-families.

(A1) The differentiable functions $f_1, f_2 : \Omega \times [0,T] \times \mathrm{H}_2 \times \mathrm{H}_3 \to \mathrm{H}_2$ and $f_3 : \Omega \times [0,T] \times \mathrm{H}_2 \times \mathrm{H}_3 \to \mathcal{L}^2(\mathrm{H}_0, \mathrm{H}_2)$ satisfy the following conditions:

- (i) The mappings f_1 and f_2 are measurable from $(\mathcal{P} \times \mathcal{B}(\mathrm{H}_2) \times \mathcal{B}(\mathrm{H}_3))$ into $(\mathrm{H}_2, \mathcal{B}(\mathrm{H}_2))$.
- (ii) The mapping f_3 is measurable from $(\mathcal{P}\times\mathcal{B}(\mathrm{H}_2)\times\mathcal{B}(\mathrm{H}_3))$ into $(\mathcal{L}^2(Q^{1/2}\mathrm{H}_1, \mathrm{H}_2), \mathcal{B}(\mathcal{L}^2(Q^{1/2}\mathrm{H}_1, \mathrm{H}_2)))$.

(A2) There exists a function $N : [0,b] \times [0,+\infty) \to [0,+\infty)$, defined as $(s,v) \to N(s,v)$, such that:

$$\begin{aligned}\mathrm{E}\|f_1(s,X(s),U(s))\|_{s_2}^p + \mathrm{E}\|f_2(s,X(s),U(s))\|_{S_2}^p + \mathrm{E}\|f_3(s,X(s),U(s))\|_{\mathcal{L}^2(\ \mathrm{S}_o,s_2)}^p \\ \leq N\left(s, \mathrm{E}\|X(s)\|_{S_2}^p\right) + N\left(s, \mathrm{E}\|U(s)\|_{S_2}^p\right) \qquad (2)\end{aligned}$$

for all $s \in [0,b]$ and all $(X,U), (\bar{X}, \bar{U}) \in L^p\left(\Omega, \mathcal{F}_s,\ \mathrm{S}_2 \times \mathrm{S}_3\right)$.

(A3) The function $N(s,v)$ is locally integrable with respect to s for each fixed $v \in [0,+\infty)$ and is continuous and non-decreasing in v for each fixed $s \in [0,b]$.

$$v(s) = \tilde{\mu}_1(b) + \tilde{\mu}_2(b) + \tilde{\mu}_3(b) \int_0^s N(r, v(r))dr$$

has a global bounded solution $\mathcal{E}(s)$ on $[0,b]$.

(A4) There is a function $K : [0,b] \times [0,+\infty) \to [0,+\infty)$ such that: for all $s \in [0,b]$ and all $(X,U), (\bar{X}, \bar{U}) \in L^p\left(\Omega, \mathcal{F}_s, \mathbb{S}_2 \times \mathbb{S}_3\right)$

$$\begin{aligned}&\mathrm{E}\|f_1(s,X(s),U(s)) - f_1(s,\bar{X}(s),\bar{U}(s))\|_{\mathbb{S}_2}^p + \mathrm{E}\|f_2(s,X(s),U(s)) - f_2(s,\bar{X}(s),\bar{U}(s))\|_{\mathbb{S}_2}^p \\ &+ \mathrm{E}\|f_3(s,X(s),U(s)) - f_3(s,\bar{X}(s),\bar{U}(s))\|_{\mathcal{L}^2(\mathrm{S}_0,S_2)}^p \\ &\leq K\left(s, \mathrm{E}\|X(s) - \bar{X}(s)\|_{\mathbb{S}_2}^p\right) + K\left(s, \mathrm{E}\|U(s) - \bar{U}(s)\|_{\mathrm{S}_3}^p\right)\end{aligned}$$

(A5) The function $K(s,v)$ is locally integrable in s for each fixed $v \in [0,+\infty)$ and continuous and non-decreasing in v for each fixed $s \in [0,b]$. Additionally, $K(s,0) = 0$. If a non-negative, continuous, bounded function $z(s)$ for $s \in [0,b]$ satisfies:

$$\begin{cases} z(s) \leq \tilde{\mu}_3(b) \int_0^s K(r, z(r))dr, & s \in [0,b] \\ z(0) = 0 \end{cases}$$

for some $\tilde{\mu}_3(b) > 0$, then $z(s) = 0$ for all $s \in [0,b]$.

(A6) $Z(0)$ and $Z'(0)$ are $\mathcal{F}_0$ -measurable H_2-valued random variables independent of B with finite $p \geq 2$ moments.

(A7) There exists a constant $M_0 > 0$ for all $0 \leq v < T$, $\quad 0 < ||\varepsilon(\varepsilon I + \Gamma_{v,T}^T)^{-1}|| < M_0$.

(A8) There exists a constant $C_p > 0$ such that for all $(y_1, u_1), (y_2, u_2) \in \mathrm{H}_2 \times \mathrm{H}_3$ and $(\omega, \tau) \in \Omega \times [0,T]$, the following inequalities hold:

$$\|f_1(\omega,\tau,y,u)\|_{\mathrm{H}_2}^p + \|f_2(\omega,\tau,y,u)\|_{\mathrm{H}_2}^p + \|f_3(\omega,\tau,y,u)\|_{\mathcal{L}^2(Q^{1/2}\mathrm{H}_1,\mathrm{H}_2)}^p \leq C_p,$$

and and $\varepsilon(\varepsilon I + \Gamma_{v,T}^T)^{-1} \underset{\lambda \to 0^+}{\longrightarrow} 0$ (converges to the zero operator as $\lambda \to 0^+$ in the strong operator topology).

Now we announce the main result. Let $\varepsilon > 0$. Initially, it is necessary to establish the existence of a unique mild solution to the equation described by (1) under the specified control:

$$\begin{aligned}
U^{\varepsilon}(\tau) &= B^*S^*(T-\tau)(\varepsilon I + \Gamma_{0,T}^{T})^{-1}\left(\mathbb{E}h - C(T)Z(0) - S(T)(Z'(0) - f_1(0, Z(0), U(0))\right) \\
&- \int_0^{\tau} B^*S^*(T-\tau)(\varepsilon I + \Gamma_{v,T}^{T})^{-1}C(T-v)f_1(v, Z(v), U(v))\mathrm{d}v \\
&- \int_0^{\tau} B^*S^*(T-\tau)(\varepsilon I + \Gamma_{v,T}^{T})^{-1}S(T-v)f_2(v, Z(v), U(v))\mathrm{d}v \\
&- \int_0^{\tau} B^*S^*(T-\tau)(\varepsilon I + \Gamma_{v,T}^{T})^{-1}S(T-v)f_3(v, Z(v), U(v)\mathrm{d}B(v) \\
&+ \int_0^{\tau} B^*S^*(T-\tau)(\varepsilon I + \Gamma_{v,T}^{T})^{-1}h(v)\mathrm{d}B(v),
\end{aligned}$$

with $\mathbb{E}\left(\int_0^T \|h(v)\|^2_{\mathcal{L}^2(Q^{1/2}\mathrm{H}_1,\mathrm{H}_2)}\, dv\right)^{\frac{p}{2}} < \infty$.

Theorem 3.2 *Assuming that conditions* $(\mathrm{A0})-(\mathrm{A6})$*, are satisfied, there is an unique mild solution* $(Z,U) \in C^p([0,T],\mathrm{H}_2) \times C^p([0,T],\mathrm{H}_3)$ *to the system described by equation* (1)*. Furthermore, possesses a continuous modification.*

Theorem 3.3 *Under the assumptions* $(\mathrm{A0})-(\mathrm{A8})$*, the stochochastic defferentiale equation* (1) *is approximately controllable, i.e. for any* $h \in \mathrm{L}^p(\Omega, \mathcal{F}_T, \mathbb{P}, \mathrm{H}_2)$,

$$\lim_{\varepsilon \longrightarrow 0_+} \mathbb{E}\,\|Z(T, Z(0), Z'(0), U^{\varepsilon}) - h\|^p_{\mathrm{H}_2} = 0.$$

Keywords: Controllability, Weak Conditions, Stopping times, Existence, Uniqueness.

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